COEFFICIENTS OF THE INFLATED EULERIAN POLYNOMIAL

JUAN S. AULI AND CARLA D. SAVAGE

ABSTRACT. It follows from work of Chung and Graham that for a certain family of polynomials $P_n(x)$ derived from the descent statistic on permutations, the sequence of nonzero coefficients of $P_{n-1}(x)$ coincides with that of the polynomial $P_n(x)/(1+x+x^2+\cdots+x^n)$.

We observed computationally that the inflated s-Eulerian polynomials, $Q_n^{(\mathbf{s})}(x)$, of which $P_n(x)$ is a special case, also have this property for many sequences \mathbf{s} . In this work we show that $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$ is a polynomial for all positive integer sequences \mathbf{s} and characterize those sequences \mathbf{s} for which the sequence of nonzero coefficients of the $Q_{n-1}^{(\mathbf{s})}(x)$ coincides with that of the polynomial $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$. In particular, we show that all nondecreasing sequences satisfy this condition.

1. Introduction

We present the solution to a problem on s-lecture hall partitions that was motivated by the paper [3] of Chung and Graham on the maxdrop statistic in permutations.

Let S_n be the set of permutations of $[n] = \{1, 2, ..., n\}$. Given a permutation $\pi = \pi_1 \pi_2 ... \pi_n$ in S_n , define $\text{Des } \pi = \{i \in [n-1] \mid \pi_i > \pi_{i-1}\}$ and $\text{des } \pi = |\text{Des } \pi|$. The following is a consequence of Theorems 4.1 and 4.2 in [3].

Proposition 1 (Chung-Graham, [3]). Let $P_n(x)$ be defined by

$$P_n(x) = \sum_{\pi \in S_n} x^{n \cdot \operatorname{des}(\pi) + \pi_n}.$$

Then

$$\frac{P_n(x)}{1+x+x^2+\cdots+x^{n-1}} = \sum_{\pi \in S_{n-1}} x^{n \cdot \operatorname{des}(\pi) + \pi_{n-1}}.$$

In particular, this means that $P_n(x)/(1+x+x^2+\cdots+x^n)$ is a polynomial and that its sequence of nonzero coefficients coincides with that of $P_{n-1}(x)$.

We will see that Proposition 1 is a special case of our main result when $\mathbf{s} = (1, 2, \dots, n)$.

Given a sequence $\mathbf{s} = (s_1, s_2, \dots, s_n)$ of positive integers, we define the **s**-lecture hall cone to be

$$C_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \dots \le \frac{\lambda_n}{s_n} \right\}.$$

The generators of this cone are $\{v_i = [0, \dots, 0, s_i, \dots, s_n] : 1 \le i \le n\}$. Its fundamental half-open parallelepiped is $\Pi_n^{(\mathbf{s})} = \{\sum_{i=1}^n \alpha_i v_i \mid 0 \le \alpha_i < 1\}$. The lattice points $C_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ are called s-lecture hall partitions [2, 9].

Date: April 4, 2015.

The generating function for the lattice points in $C_n^{(s)}$ can be computed from its fundamental parallelepiped and generators as

(1.1)
$$\sum_{\substack{\lambda \in C^{(\mathbf{s})} \cap \mathbb{Z}^n \\ 1}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} = \frac{\sum_{\substack{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n \\ \prod_{i=1}^n (1 - x_i^{s_i} \cdots x_n^{s_n})}}{\prod_{i=1}^n (1 - x_i^{s_i} \cdots x_n^{s_n})}.$$

Setting $x_1 = \ldots = x_{n-1} = 1$ and $x_n = x$ in (1.1) gives

(1.2)
$$\sum_{\lambda \in C_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n} = \frac{\sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n}}{(1 - x^{s_n})^n}.$$

Define $Q_n^{(\mathbf{s})}(x)$ by

$$Q_n^{(\mathbf{s})}(x) = \sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n}.$$

We will show in Section 3 that $Q_n^{(s)}(x)$ is the *inflated s-Eulerian polynomial* associated to $C_n^{(s)}$, introduced in [8].

We observed that for particular (infinite) sequences \mathbf{s} and positive integers n the sequence of nonzero coefficients of $Q_{n-1}^{(\mathbf{s})}(x)$ coincides with that of the polynomial $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$. This surprising fact lead us to consider the problem of characterizing all sequences \mathbf{s} satisfying this condition for all n. We call these sequences *contractible*.

Note that the polynomiality of $Q_n^{(s)}(x)/(1+x+\cdots+x^{s_n-1})$ is not evident from the definition of the $Q_n^{(s)}(x)$.

Example. Let **s** be the Fibonacci sequence. Then the nth inflated **s**-Eulerian polynomials for the first few n are as follows:

$$\begin{split} Q_5^{(1,1,2,3,5)}(x) &= \mathbf{1} x^{11} + \mathbf{1} x^{10} + \mathbf{2} x^9 + \mathbf{4} x^8 + \mathbf{4} x^7 + \mathbf{4} x^6 + \mathbf{4} x^5 + \mathbf{4} x^4 + \mathbf{2} x^3 + \mathbf{2} x^2 + \mathbf{1} x + \mathbf{1}, \\ Q_4^{(1,1,2,3)}(x) &= \mathbf{1} x^4 + \mathbf{1} x^3 + \mathbf{2} x^2 + \mathbf{1} x + \mathbf{1}, \\ Q_3^{(1,1,2)}(x) &= \mathbf{1} x + \mathbf{1}, \text{ and} \\ Q_2^{(1,1)}(x) &= \mathbf{1}. \end{split}$$

On the other hand, the polynomials $Q_n^{(s)}(x)/(1+x+\cdots+x^{s_n-1})$ for the first few n are:

$$\begin{split} \frac{Q_6^{(1,1,2,3,5,8)}(x)}{1+x+\cdots+x^7} &= \mathbf{1}x^{18} + \mathbf{1}x^{16} + \mathbf{2}x^{15} + \mathbf{4}x^{13} + \mathbf{4}x^{12} + \mathbf{4}x^{10} + \mathbf{4}x^8 + \mathbf{4}x^7 + \mathbf{2}x^5 + \mathbf{2}x^4 + \mathbf{1}x^2 + \mathbf{1}, \\ \frac{Q_5^{(1,1,2,3,5)}(x)}{1+x+\cdots+x^4} &= \mathbf{1}x^7 + \mathbf{1}x^5 + \mathbf{2}x^4 + \mathbf{1}x^2 + \mathbf{1}, \\ \frac{Q_4^{(1,1,2,3)}(x)}{1+x+x^2} &= \mathbf{1}x^2 + \mathbf{1}, \text{ and} \\ \frac{Q_3^{(1,1,2)}(x)}{1+x} &= \mathbf{1}. \end{split}$$

In this paper, we prove that all nondecreasing sequences are contractible. Moreover, we characterize contractible sequences as follows.

Theorem 2. A sequence **s** of positive integers is contractible if and only if either $(s_n)_{n=3}^{\infty}$ is nondecreasing, or there exists $N \geq 3$ such that $(s_n)_{n=N}^{\infty}$ is nondecreasing, $s_N = s_{N-1} - 1$ and $s_j = 1$ for j = 1, 2, ..., N-2.

Our work relies on a combinatorial interpretation of $Q_n^{(s)}(x)$ due to Pensyl and Savage [8], which involves s-inversion sequences.

The paper is organized as follows. In Section 2, we provide the historical background of this problem. In particular, we explain how $Q_n^{(\mathbf{s})}(x)$ is related to the s-lecture hall partitions introduced by Bousquet-Mélou and Eriksson [2]. Section 3 presents Pensyl and Savage's description of $Q_n^{(\mathbf{s})}(x)$ in terms of certain statistics on s-inversion sequences. In Section 4, we provide a combinatorial characterization of $Q_n^{(\mathbf{s})}(x)/(1+x+\cdots+x^{s_n-1})$ that, in particular, proves its polynomiality. In Section 5, we introduce a lemma (Lemma 10), which is the main ingredient in the proof of Theorem 2. We then use this lemma to show that all nondecreasing sequences are contractible. Finally, we conclude the proof of Theorem 2 in Section 6.

2. Lecture hall partitions and $Q_n^{(\mathbf{s})}(x)$

Lecture hall partitions were introduced in 1997 by Bousquet-Mélou and Eriksson [1]. They defined a lecture hall partition into n parts as a partition $(\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfying the inequality

$$(2.1) 0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \dots \le \frac{\lambda_n}{n}.$$

Note that this condition ensures that $\lambda_i \leq \lambda_{i+1}$ for $1 \leq i \leq n-1$. Furthermore, if we order the parts $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of a lecture hall partition from left to right in a diagram (see Figure 2.1 for an example), we obtain a figure resembling a lecture hall. Indeed, (2.1) is a sufficient condition to allow students (A, B, C and D in the picture) in each row (part) to see the professor (O in the picture). This fact led to the name of these restricted partitions.

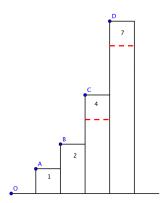


FIGURE 2.1. Lecture hall partition 14 = 1 + 2 + 4 + 7.

Example. The partition 14 = 1 + 2 + 4 + 7 is a lecture hall partition because $\frac{1}{1} \le \frac{2}{2} \le \frac{4}{3} \le \frac{7}{4}$. Note that a partition into 4 parts such that $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 4$ must satisfy $\frac{4}{3} \le \frac{\lambda_4}{4}$ in order to be a lecture hall partition, so necessarily, $\lambda_4 \ge 6$. Thus, 12 = 1 + 2 + 4 + 5 is not a lecture hall partition.

The most remarkable result about lecture hall partitions is the Lecture Hall Theorem, due to Bousquet-Mélou and Eriksson [1]. This theorem relates lecture hall partitions to partitions into bounded odd parts.

Theorem 3 (Bousquet-Mélou and Eriksson). For n fixed, the generating function for the number of lecture hall partitions of N into n parts, LH(N,n) is given by

$$\sum_{N=0}^{\infty} LH(N,n)q^{N} = \prod_{i=1}^{n} \frac{1}{1 - q^{2i-1}}.$$

That is, it coincides with the generating function of integer partitions into odd parts less than 2n.

The reader interested in a bijective proof of this result is referred to [7] and [10].

Since lecture hall partitions have distinct parts, we may think of this result as a finite version of Euler's theorem asserting that the number of partitions of N into distinct parts coincides with the number of partitions of N into odd parts.

A natural generalization of lecture hall partitions is to consider sequences $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of positive integers such that

$$0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \dots \le \frac{\lambda_n}{s_n},$$

where $\mathbf{s} = (s_1, s_2, \dots, s_n)$ is a sequence of positive integers. These \mathbf{s} -lecture hall partitions were introduced by Bousquet-Mélou and Eriksson for nondecreasing \mathbf{s} in [2] and by Savage and others for arbitrary positive integer sequences in [5, 4, 9, 6].

Definition. Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be a sequence of positive integers. We define the **s**-lecture hall cone to be

$$C_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \dots \le \frac{\lambda_n}{s_n} \right\}.$$

This cone has generators $\{v_i = [0, \dots, 0, s_i, \dots, s_n] : 1 \le i \le n\}$ and its fundamental half-open parallelepiped is given by

$$\Pi_n^{(\mathbf{s})} = \left\{ \sum_{i=1}^n \alpha_i v_i \mid 0 \le \alpha_i < 1 \right\}.$$

Example. Let $\mathbf{r} = (3,2)$, then there are 6 \mathbf{r} -lecture hall partitions in the fundamental half-open parallelepiped of \mathbf{r} . Indeed, $\Pi_2^{(3,2)} \cap \mathbb{Z}^2 = \{(0,0),(0,1),(1,1),(1,2),(2,2),(2,3)\}$ (see Figure 2.4). Note that in general, for a positive sequence \mathbf{s} it is true that $\left|\Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n\right| = s_1 s_2 \cdots s_n$.

Note that each s-lecture hall partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is an element of $C_n^{(\mathbf{s})}$ when considered as an ordered *n*-dimensional vector. For instance, (1,2) and (2,1) are points of the (5,2)-lecture hall cone and, by definition, they are distinct (5,2)-lecture hall partitions of 3 (see Figure 2.2). However, if \mathbf{s} is nondecreasing, then the points in $C_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ are actually partitions on the traditional sense, in that the sequence λ is nondecreasing, so no permutation of its coordinates appears more than once.

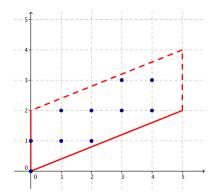


FIGURE 2.2. Fundamental half-open parallelepiped of $C_2^{(5,2)}$.

Definition. Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be a sequence of positive integers. We define the *nth inflated* \mathbf{s} -Eulerian polynomial by

$$Q_n^{(\mathbf{s})}(x) = \sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n}.$$

Example. The sequence $\mathbf{s} = (5,3)$ has \mathbf{s} -lecture hall cone $C_2^{(5,3)} = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid 0 \leq \frac{\lambda_1}{5} \leq \frac{\lambda_2}{3} \right\}$, with generators [5,3] and [0,3]. There are $s_1 \cdot s_2 = 15$ s-lecture hall partitions in the fundamental half-open parallelepiped (see Figure 2.3), namely, $\Pi_2^{(5,3)} \cap \mathbb{Z}^2$ is given by

$$\{(0,0),(0,1),(1,1),(0,2),(1,2),(2,2),(3,2),(1,3),(2,3),(3,3),(4,3),(2,4),(3,4),(4,4),(4,5)\}.$$

Therefore, the inflated s-Eulerian polynomial is given by

$$Q_2^{(5,3)}(x) = \sum_{(\lambda_1,\lambda_2)\in\Pi_2^{(5,3)}\cap\mathbb{Z}^2} x^{\lambda_2} = 1 + 2x + 4x^2 + 4x^3 + 3x^4 + x^5.$$

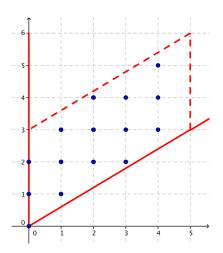


FIGURE 2.3. The cone $C_2^{(5,3)}$ and its fundamental half-open parallelepiped.

The following example illustrates that the inflated s-Eulerian polynomial depends not only on the positive integers in a sequence, but also on the order in which they appear.

Example. Let $\mathbf{s} = (2,3)$ and $\mathbf{r} = (3,2)$. These sequences have lecture hall cones

$$C_2^{(\mathbf{s})} = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid 0 \le \frac{\lambda_1}{2} \le \frac{\lambda_2}{3} \right\} \text{and } C_2^{(\mathbf{r})} = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid 0 \le \frac{\lambda_1}{3} \le \frac{\lambda_2}{2} \right\},$$

respectively (see Figure 2.4). Since $s_1 \cdot s_2 = 6 = r_1 \cdot r_2$, there are 6 s-lecture hall (resp. r-lecture hall) partitions in the fundamental parallelepiped $\Pi_2^{(\mathbf{s})}$ (resp. $\Pi_2^{(\mathbf{r})}$), which implies that $Q_2^{(\mathbf{s})}(1) = 6 = Q_2^{(\mathbf{r})}(1)$. However, the inflated Eulerian polynomials of these sequences do not coincide, for $Q_2^{(\mathbf{s})}(x) = 1 + x + 2x^2 + x^3 + x^4$ and $Q_2^{(\mathbf{r})}(x) = 1 + 2x + 2x^2 + x^3$.

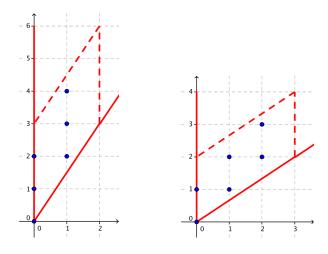


Figure 2.4. Comparison between $\Pi_2^{(2,3)}$ and $\Pi_2^{(3,2)}$.

3. An alternative description of $Q_n^{(\mathbf{s})}(x)$

The definition of the inflated s-Eulerian polynomial is very intuitive. However, we now introduce an alternative description of $Q_n^{(s)}(x)$, due to Pensyl and Savage [8], that is more convenient for our purposes.

Definition. Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be a finite sequence of positive integers. We define the sinversion sequences as

$$I_n^{(\mathbf{s})} = \{ \mathbf{e} = (e_1, e_2, \dots, e_n) \in \mathbb{Z}^n \mid 0 \le e_i < s_i \text{ for } 1 \le i \le n \}.$$

Given $\mathbf{e} \in I_n^{(\mathbf{s})}$, we say $1 \le i < n$ is an ascent of \mathbf{e} if $\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}$. We say 0 is an ascent of \mathbf{e} if $e_1 > 0$.

It is customary to denote the set of ascents of $\mathbf{e} \in I_n^{(\mathbf{s})}$ by $\mathrm{Asc}(\mathbf{e})$ and its cardinality by $\mathrm{asc}(\mathbf{e})$. We denote the collection $\left\{\mathrm{Asc}(\mathbf{e}) \mid \mathbf{e} \in I_n^{(\mathbf{s})}\right\}$ by $\mathrm{Asc}_n^{(\mathbf{s})}$.

Example. Consider the sequence $\mathbf{s}=(1,1,2,2)$. By definition, the **s**-inversion sequences are given by $I_4^{(1,1,2,2)}=\{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,0,1,1)\}$. Note that $3\notin \mathrm{Asc}(0,0,1,1)$, because $\frac{1}{2}\not<\frac{1}{2}$. However, $\frac{0}{1}<\frac{1}{2}$ implies that $2\in \mathrm{Asc}(0,0,1,1)$.

We now state Pensyl and Savage's result. It leads to a description of $Q_n^{(s)}(x)$ in terms of the ascents of the s-inversion sequences $I_n^{(s)}$.

Theorem 4 (Pensyl-Savage, [8]). Let s be a sequence of positive integers. Then

(3.1)
$$\sum_{\lambda \in C_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n} = \frac{\sum_{\mathbf{e} \in I_n^{(\mathbf{s})}} x^{\operatorname{asc}(\mathbf{e}) - e_n}}{(1 - x^{s_n})^n}.$$

Combining Theorem 4 with (1.2) we get the following corollary.

Corollary 5. Let s be a sequence of positive integers, then

$$Q_n^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in I_n^{(\mathbf{s})}} x^{s_n \operatorname{asc}(\mathbf{e}) - e_n}.$$

Example. Consider the sequence $\mathbf{s} = (1, 2, 3)$. The s-inversion sequences

$$I_3^{(1,2,3)} = \{(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,1,2)\}$$

have ascents $Asc_3^{(1,2,3)} = \{\emptyset, \{2\}, \{2\}, \{1\}, \{1\}, \{1,2\}\}, \text{ respectively. By Corollary 5},$

$$\begin{split} Q_3^{(1,2,3)}(x) &= x^{3(0)-0} + x^{3(1)-1} + x^{3(1)-2} + x^{3(1)-0} + x^{3(1)-1} + x^{3(2)-2} \\ &= x^0 + x^2 + x + x^3 + x^2 + x^4 \\ &= x^4 + x^3 + 2x^2 + x + 1. \end{split}$$

4. Contractible sequences

In this section, we introduce the notion of contractibility for a sequence **s** of positive integers. We begin by showing that for any sequence **s** and any n, the expression $Q_n^{(s)}(x)/(1+x+\cdots+x^{s_n-1})$ is a polynomial with integer coefficients. Furthermore, we provide a very convenient combinatorial description of this polynomial.

Given a sequence $\mathbf{e} = (e_1, e_2, \dots, e_{n-1}) \in I_{n-1}$ and $0 \le k < s_n$, we denote $(e_1, e_2, \dots, e_{n-1}, k) \in I_n$ by (\mathbf{e}, k) .

Theorem 6. Let **s** be a sequence of positive integers, then

$$\frac{Q_n^{(\mathbf{s})}(x)}{1+x+\dots+x^{s_n-1}} = \sum_{\mathbf{e}\in I_{n-1}^{(\mathbf{s})}} x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor}.$$

Proof. Let **s** be a sequence of positive integers and $\mathbf{e} \in I_{n-1}^{(\mathbf{s})}$. Let a be the smallest positive integer such that $\frac{e_{n-1}}{s_{n-1}} < \frac{a}{s_n}$, then $a-1 \le \frac{s_n e_{n-1}}{s_{n-1}} < a$, so $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = a-1$. Note that $\frac{e_{n-1}}{s_{n-1}} < 1$ implies that $a \le s_n$. Write

$$\left(\sum_{k=0}^{s_n-1} x^k\right) x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor} = \left(\sum_{k=0}^{s_n-1} x^k\right) x^{s_n \operatorname{asc}(\mathbf{e}) - a + 1}$$

$$= \left(\sum_{k=0}^{a-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a - k - 1)}\right) + \left(\sum_{k=a}^{s_n-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a - k - 1)}\right).$$
(4.1)

If $a = s_n$, the rightmost sum is empty. Since $\frac{e_{n-1}}{s_{n-1}} \ge \frac{k}{s_n}$ for $0 \le k \le a - 1$,

$$\sum_{k=0}^{a-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a-k-1)} = x^{s_n \operatorname{asc}(\mathbf{e}) - 0} + x^{s_n \operatorname{asc}(\mathbf{e}) - 1} + \dots + x^{s_n \operatorname{asc}(\mathbf{e}) - (a-1)}$$
$$= x^{s_n \operatorname{asc}(\mathbf{e}, 0) - 0} + x^{s_n \operatorname{asc}(\mathbf{e}, 1) - 1} + \dots + x^{s_n \operatorname{asc}(\mathbf{e}, a-1) - (a-1)}.$$

Similarly, the fact that $\frac{e_{n-1}}{s_{n-1}} < \frac{k}{s_n}$ for $a \le k \le s_n - 1$ implies that

$$\sum_{k=a}^{s_n-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a-k-1)} = x^{s_n \operatorname{asc}(\mathbf{e}) + 1} + x^{s_n \operatorname{asc}(\mathbf{e}) + 2} + \dots + x^{s_n \operatorname{asc}(\mathbf{e}) + s_n - a}$$

$$= x^{s_n(1 + \operatorname{asc}(\mathbf{e})) - (s_n - 1)} + x^{s_n(1 + \operatorname{asc}(\mathbf{e})) - (s_n - 2)} + \dots + x^{s_n(1 + \operatorname{asc}(\mathbf{e})) - a}$$

$$= x^{s_n \operatorname{asc}(\mathbf{e}, s_n - 1) - (s_n - 1)} + x^{s_n \operatorname{asc}(\mathbf{e}, s_n - 2) - (s_n - 2)} + \dots + x^{s_n \operatorname{asc}(\mathbf{e}, a) - a}.$$

By (4.1), we deduce that for each $\mathbf{e} \in I_{n-1}^{(\mathbf{s})}$,

$$\left(\sum_{k=0}^{s_n-1} x^k\right) x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor} = \sum_{k=0}^{s_n-1} x^{s_n \operatorname{asc}(\mathbf{e},k) - k},$$

so we may write

$$(1+x+\cdots+x^{s_n-1})\sum_{\mathbf{e}\in I_{n-1}^{(\mathbf{s})}}x^{s_n\mathrm{asc}(\mathbf{e})-\left\lfloor\frac{s_ne_{n-1}}{s_{n-1}}\right\rfloor}=\sum_{\mathbf{e}\in I_n^{(\mathbf{s})}}x^{s_n\mathrm{asc}(\mathbf{e})-e_n}=Q_n^{(\mathbf{s})}(x).$$

Note that the rightmost equality follows from Corollary 5.

The following corollary is an immediate consequence of Theorem 6.

Corollary 7 (Chung-Graham, [3]). Let $Q_n^{(1,2,\ldots,n)}(x)$ be the inflated **n**-Eulerian polynomial, then

$$\frac{Q_n^{(1,2,\dots,n)}(x)}{1+x+\dots+x^{n-1}} = \sum_{\mathbf{e}\in I_{n-1}} x^{n\cdot\operatorname{asc}(\mathbf{e})-e_{n-1}}.$$

To see that Corollary 7 is equivalent to Proposition 1, consider the mapping $\phi: S_n \to I_n$ defined by $\phi(\pi) = (e_1, \dots, e_n)$ where $e_i = |\{j > 0 \mid j < i \text{ and } \pi_j > \pi_i\}|$. Then $\operatorname{Des} \pi = \operatorname{Asc}(\phi(\pi))$ and $e_n = n - \pi_n$.

It follows from Corollaries 5 and 7 imply that the sequence of nonzero coefficients of $Q_{n-1}^{(1,2,\dots,n-1)}(x)$ and $Q_n^{(1,2,\dots,n)}(x)/(1+x+\dots+x^{n-1})$ coincide for all n. Computational trials show that this event is common among positive sequences \mathbf{s} , at least for small n. This fact motivates the next definition.

Definition. If $n \geq 3$, we say a positive sequence **s** is *n*-contractible if the sequence of nonzero coefficients of $Q_{n-1}^{(\mathbf{s})}(x)$ and $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$ coincide. If a sequence **s** is *n*-contractible for $n \geq 3$, then **s** is contractible.

Example. Let **s** be the Fibonacci sequence. Using Corollary 5, it is possible to compute $Q_{n-1}^{(\mathbf{s})}(x)$ and $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$ for the first few n (see Example on page 1), and verify that the Fibonacci sequence is n-contractible for n=3,4,5,6.

The following corollary shows that all positive constant sequences are contractible. Furthermore, the polynomials $Q_{n-1}^{(\mathbf{s})}(x)$ and $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$ coincide if \mathbf{s} is constant.

Corollary 8. Let s be a positive constant sequence, then for all $n \geq 2$

$$Q_{n-1}^{(\mathbf{s})}(x) = \frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \dots + x^{s_n - 1}}.$$

Proof. Say $s_n = k$ for all n, then Theorems 4 and 6 imply that

$$\frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \dots + x^{s_n - 1}} = \sum_{\mathbf{e} \in I_{n - 1}^{(\mathbf{s})}} x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n - 1}}{s_{n - 1}} \right\rfloor} = \sum_{\mathbf{e} \in I_{n - 1}^{(\mathbf{s})}} x^{k \cdot \operatorname{asc}(\mathbf{e}) - e_{n - 1}} = Q_{n - 1}^{(\mathbf{s})}(x).$$

5. The case of nondecreasing sequences

In this section, we prove that all nondecreasing sequences are contractible. As we mentioned before, this case is of particular interest because for such \mathbf{s} , the set $C_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ consists the \mathbf{s} -lecture hall partitions with n parts.

By Corollary 5 and Theorem 6, we know that for any sequence s of positive integers

$$Q_{n-1}^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1}} \quad \text{and} \quad \frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \dots + x^{s_n - 1}} = \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{s_n \mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor}.$$

Therefore, an *n*-contractible sequence **s** is one such that for all $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$,

(5.1)
$$s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1} \quad \text{if and only if} \quad s_n\operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n\operatorname{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

We now show that the forward direction of (5.1) holds for arbitrary positive sequences **s** and any $n \ge 3$.

Lemma 9. Let **s** be a sequence of positive integers and $n \geq 3$. Suppose **e** and $\overline{\mathbf{e}}$ are elements of $I_{n-1}^{(\mathbf{s})}$ such that

$$s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1},$$

then

$$s_n \operatorname{asc}(\mathbf{e}) - \left| \frac{s_n e_{n-1}}{s_{n-1}} \right| = s_n \operatorname{asc}(\overline{\mathbf{e}}) - \left| \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right|.$$

Proof. Let $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ and suppose that $s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}$. Without loss of generality, say $e_{n-1} \geq \overline{e}_{n-1}$ and write $s_{n-1}\mathrm{asc}(\mathbf{e}) = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) + (e_{n-1} - \overline{e}_{n-1})$. This means that $e_{n-1} - \overline{e}_{n-1} \equiv 0 \mod s_{n-1}$, but

$$0 \le e_{n-1} - \overline{e}_{n-1} < s_{n-1} - \overline{e}_{n-1} \le s_{n-1},$$

so it must be the case that $e_{n-1} = \overline{e}_{n-1}$ and this forces $asc(\mathbf{e}) = asc(\overline{\mathbf{e}})$. Hence,

$$s_n \operatorname{asc}(\mathbf{e}) - \left| \frac{s_n e_{n-1}}{s_{n-1}} \right| = s_n \operatorname{asc}(\overline{\mathbf{e}}) - \left| \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right|.$$

By (5.1) and Lemma 9, we now have the following characterization of n-contractible sequences.

Lemma 10. Let **s** be a sequence of positive integers and $n \ge 3$. Then **s** is n-contractible if and only if whenever $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ are such that

$$s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \operatorname{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor,$$

then

$$s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}.$$

We now show that $s_n \ge s_{n-1}$ implies that **s** is *n*-contractible, and hence nondecreasing sequences are contractible.

Proposition 11. Let **s** be a sequence of positive integers and $n \geq 3$. If $s_n \geq s_{n-1}$, then **s** is n-conractible. Therefore, nondecreasing sequences are contractible.

Proof. Let $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ be such that

(5.2)
$$s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \operatorname{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

Since $0 \le e_{n-s}, \overline{e}_{n-1} < s_{n-1}$, we know that the inequality $0 \le \frac{s_n e_{n-1}}{s_{n-1}}, \frac{s_n \overline{e}_{n-1}}{s_{n-1}} < s_n$ holds, so

$$0 \le \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor, \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor \le s_n - 1.$$

Note that (5.2) implies that $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor$ and $\left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$ are congruent modulo s_n , so we deduce that

$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor,$$

and therefore, $\operatorname{asc}(\mathbf{e}) = \operatorname{asc}(\overline{\mathbf{e}})$. Assume that $s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} \neq s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}$, then it must be that $e_{n-1} \neq \overline{e}_{n-1}$. Say $e_{n-1} > \overline{e}_{n-1}$, then

$$\left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor \le \left\lfloor \frac{s_n (\overline{e}_{n-1} + 1)}{s_{n-1}} \right\rfloor \le \dots \le \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

Thus, $\left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} + \frac{s_n}{s_{n-1}} \right\rfloor \geq \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} + 1 \right\rfloor = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor + 1$, which is a contradiction. We conclude that $s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}$. The result then follows from Lemma 10. \square

Remark. Note that in the preceding proof the condition $s_n \geq s_{n-1}$ is not used to conclude that

$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

Therefore, for any s, if $e, \overline{e} \in I_{n-1}^{(s)}$ are such that

$$s_n \operatorname{asc}(\mathbf{e}) - \left| \frac{s_n e_{n-1}}{s_{n-1}} \right| = s_n \operatorname{asc}(\overline{\mathbf{e}}) - \left| \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right|,$$

then
$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$$
 and $\operatorname{asc}(\mathbf{e}) = \operatorname{asc}(\overline{\mathbf{e}})$.

6. A CHARACTERIZATION OF CONTRACTIBLE SEQUENCES

Although the criterion for contractibility provided by Proposition 11 requires relatively weak conditions on \mathbf{s} , there do exist noncontractible sequences. Indeed, the next example exhibits an infinite family of noncontractible sequences.

Example. Consider the finite sequence (1,7,2). Corollary 5 and Theorem 6 imply that

$$Q_2^{(1,7)}(x) = \mathbf{1}x^6 + \mathbf{1}x^5 + \mathbf{1}x^4 + \mathbf{1}x^3 + \mathbf{1}x^2 + \mathbf{1}x + \mathbf{1}$$
, and
$$\frac{Q_3^{(1,7,2)}(x)}{1+x} = \mathbf{3}x^2 + \mathbf{3}x + \mathbf{1}.$$

Therefore, any **s** such that $(s_1, s_2, s_3) = (1, 7, 2)$ is not contractible.

By Proposition 11, we know that all nondecreasing sequences are contractible. Furthermore, this proposition implies that if **s** is a sequence of positive integers such that $(s_n)_{n=3}^{\infty}$ is nondecreasing, then **s** is contractible. The next example shows that this sufficient criterion is not necessary.

Example. Let **s** be the sequence defined by

$$s_n = \begin{cases} 1 & \text{if } n \in \{1, 2\}, \\ 3 & \text{if } n = 3 \text{ and,} \\ 2 & \text{if } n \ge 4. \end{cases}$$

Using Corollary 5, we find that

$$Q_3^{(\mathbf{s})}(x) = \mathbf{1}x^2 + \mathbf{1}x + \mathbf{1}$$
, and $Q_2^{(\mathbf{s})}(x) = \mathbf{1}$.

On the other hand, Theorem 6 implies that

$$\frac{Q_4^{(\mathbf{s})}(x)}{1+x} = \mathbf{1}x^2 + \mathbf{1}x + \mathbf{1}, \text{ and}$$
$$\frac{Q_3^{(\mathbf{s})}(x)}{1+x+x^2} = \mathbf{1}.$$

Suppose that n > 4, then $s_n \ge s_{n-1}$ and it follows from Proposition 11 that **s** is *n*-contractible. We see that although $s_4 < s_3$, the sequence **s** is contractible.

We now conclude the proof of Theorem 2, which we recall:

Theorem 2. A sequence **s** of positive integers is contractible if and only if either $(s_n)_{n=3}^{\infty}$ is nondecreasing, or there exists $N \geq 3$ such that $(s_n)_{n=N}^{\infty}$ is nondecreasing, $s_N = s_{N-1} - 1$ and $s_j = 1$ for j = 1, 2, ..., N-2.

Our strategy is to exploit the characterization of n-contractible sequences provided by Lemma 10. In order to do this, we need the following lemma.

Lemma 12. Let **s** be a sequence of positive integers. Suppose $n \ge 3$ is such that $s_n \le s_{n-1} - 1$. If there exists $1 \le j \le n - 2$ such that $s_j > 1$, then **s** is not n-contractible.

Proof. Note that $\mathbf{e} = (0, 0, \dots, 0, 1)$ and $\overline{\mathbf{e}} = (0, \dots, 0, 1, 0, \dots 0)$ (1 in the jth entry) are elements of $I_{n-1}^{(\mathbf{s})}$ such that $\operatorname{asc}(\mathbf{e}) = 1 = \operatorname{asc}(\overline{\mathbf{e}})$ and $e_{n-1} = 1 \neq 0 = \overline{e}_{n-1}$. Given that $\left|\frac{s_n e_{n-1}}{s_{n-1}}\right| = 0 = \left|\frac{s_n \overline{e}_{n-1}}{s_{n-1}}\right|$,

$$s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} \neq s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1} \text{ and } s_n\operatorname{asc}(\mathbf{e}) - \left|\frac{s_ne_{n-1}}{s_{n-1}}\right| = s_n\operatorname{asc}(\overline{\mathbf{e}}) - \left|\frac{s_n\overline{e}_{n-1}}{s_{n-1}}\right|.$$

The result then follows from Lemma 10.

By Lemma 12, it only remains to consider sequences of the form $(1, 1, ..., 1, s_{N-1}, s_N, ...)$, where $N \geq 3$, $s_N \leq s_{N-1} - 1$ and $(s_n)_{n=N}^{\infty}$ is nondecreasing. Indeed, in order to conclude the proof of Theorem 2, it suffices to show that if $n \geq 3$ and **s** is a sequence such that $s_1 = s_2 = \cdots = s_{n-2} = 1$ and $s_n \leq s_{n-1} - 1$, then **s** is *n*-contractible if and only if $s_n = s_{n-1} - 1$.

Proof of Theorem 2. Let $n \ge 3$ and suppose **s** is a sequence such that $s_1 = s_2 = \cdots = s_{n-2} = 1$ and $s_n \le s_{n-1} - 1$.

Assume $s_n = s_{n-1} - 1$. Let $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ be such that $s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \operatorname{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$. By the remark following Proposition 11, we know that $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$ and $\operatorname{asc}(\mathbf{e}) = \operatorname{asc}(\overline{\mathbf{e}})$. Write

$$\left| \frac{s_n e_{n-1}}{s_{n-1}} \right| = \left| \frac{(s_{n-1} - 1)e_{n-1}}{s_{n-1}} \right| = e_{n-1} + \left| \frac{-e_{n-1}}{s_{n-1}} \right|.$$

This means that

$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \begin{cases} 0 & \text{if } e_{n-1} = 0, \\ e_{n-1} - 1 & \text{otherwise.} \end{cases}$$

Hence, if $e_{n-1} \neq \overline{e}_{n-1}$, then (without loss of generality) $e_{n-1} = 0$ and $\overline{e}_{n-1} = 1$. This means that $\mathbf{e} = (0, 0, \dots, 0)$ and $\overline{\mathbf{e}} = (0, \dots, 0, 1)$, but then $\operatorname{asc}(\mathbf{e}) = 0 \neq 1 = \operatorname{asc}(\overline{\mathbf{e}})$ (a contradiction). We deduce that $e_{n-1} = \overline{e}_{n-1}$ and consequently,

$$s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}.$$

Using Lemma 10, we conclude that **s** is *n*-contractible if $s_n = s_{n-1} - 1$.

Now, suppose $s_n < s_{n-1} - 1$. Say $s_n = s_{n-1} - l$ for $2 \le l < s_{n-1}$. If $0 \le m < s_{n-1}$, then

(6.1)
$$\left| \frac{s_n m}{s_{n-1}} \right| = \left| \frac{(s_{n-1} - l)m}{s_{n-1}} \right| = m + \left| \frac{-lm}{s_{n-1}} \right|.$$

Note that $\left\lfloor \frac{-l(s_{n-1}-1)}{s_{n-1}} \right\rfloor = -l + \left\lfloor \frac{l}{s_{n-1}} \right\rfloor = -l \le -2$. Thus, we may choose k to be the smallest integer in $\{2, 3, \ldots, s_{n-1} - 1\}$ such that $\left\lfloor \frac{-lk}{s_{n-1}} \right\rfloor \le -2$. Since $k-1 \ge 1$, it must be that $\left\lfloor \frac{-l(k-1)}{s_{n-1}} \right\rfloor = -1$. By (6.1), we may write

$$\left[\frac{s_n(k-1)}{s_{n-1}} \right] = (k-1) + (-1) = k-2.$$

Now,

$$\left\lfloor \frac{-lk}{s_{n-1}} \right\rfloor = \left\lfloor \frac{-l(k-1)}{s_{n-1}} - \frac{l}{s_{n-1}} \right\rfloor \ge \left\lfloor \frac{-l(k-1)}{s_{n-1}} \right\rfloor + \left\lfloor \frac{-l}{s_{n-1}} \right\rfloor = (-1) + (-1) = -2.$$

Therefore, $\left\lfloor \frac{-lk}{s_{n-1}} \right\rfloor = -2$ and it follows from (6.1) that $\left\lfloor \frac{s_nk}{s_{n-1}} \right\rfloor = k-2$. Let \mathbf{e} and $\overline{\mathbf{e}}$ in $I_{n-1}^{(\mathbf{s})}$ be given by $\mathbf{e} = (0, \dots, 0, k)$ and $\overline{\mathbf{e}} = (0, \dots, 0, k-1)$, then $\operatorname{asc}(\mathbf{e}) = 1 = \operatorname{asc}(\overline{\mathbf{e}})$ and $\left\lfloor \frac{s_ne_{n-1}}{s_{n-1}} \right\rfloor = k-2 = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor$,

$$s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} \neq s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1} \text{ and } s_n\operatorname{asc}(\mathbf{e}) - \left|\frac{s_ne_{n-1}}{s_{n-1}}\right| = s_n\operatorname{asc}(\overline{\mathbf{e}}) - \left|\frac{s_n\overline{e}_{n-1}}{s_{n-1}}\right|.$$

By Lemma 10, we deduce that **s** is not n-contractible if $s_n < s_{n-1} - 1$.

References

- [1] M. Bousquet-Mélou and K. Eriksson. Lecture hall partitions. The Ramanujan Journal, 1(1):101–111, 1997.
- [2] M. Bousquet-Mélou and K. Eriksson. Lecture hall partitions II. The Ramanujan Journal, 1(2):165–185, 1997.
- [3] F. Chung and R. Graham. Inversion-descent polynomials for restricted permutations. J. Comb. Theory Ser. A, 120(2):366–378, 2013.
- [4] S. Corteel, S. Lee, and C. D. Savage. Enumeration of sequences constrained by the ratio of consecutive parts. Sém. Lothar. Combin., 54A:Art. B54Aa, 12, 2005/07.
- [5] S. Corteel and C. D. Savage. Anti-lecture hall compositions. Discrete Math., 263(1-3):275-280, 2003.
- [6] S. Corteel, C. D. Savage, and A. V. Sills. Lecture hall sequences, q-series, and asymmetric partition identities. In *Partitions, q-series, and modular forms*, volume 23 of *Dev. Math.*, pages 53–68. Springer, New York, 2012.
- [7] N. Eriksen. A simple bijection between lecture hall partitions and partitions into odd integers. In Formal Power Series and Algebraic Combinatorics, 2002.
- [8] T. W. Pensyl and C. D. Savage. Rational lecture hall polytopes and inflated Eulerian polynomials. *The Ramanujan Journal*, 31(1-2):97–114, 2013.
- [9] C. D. Savage and M. J. Schuster. Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences. *J. Combin. Theory Ser. A*, 119(4):850–870, 2012.
- [10] A. J. Yee. On the combinatorics of lecture hall partitions. The Ramanujan Journal, 5(3):247-262, 2001.

Department of Mathematics, San Francisco State University, San Francisco, CA, 94132

E-mail address: jauly@mail.sfsu.edu

DEPARTMENT OF COMPUTER SCIENCE, NORTH CAROLINA STATE UNIVERSITY, RALEIGH NC, 27695-8206

E-mail address: savage@ncsu.edu